Synchronous chaos in coupled map lattices with small-world interactions

Prashant M. Gade* and Chin-Kun Hu†
Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan
(Received 14 March 2000; revised manuscript received 27 July 2000)

In certain physical situations, extensive interactions arise naturally in systems. We consider one such situation, namely, small-world couplings. We show that, for a fixed fraction of nonlocal couplings, synchronous chaos is always a stable attractor in the thermodynamic limit. We point out that randomness helps synchronization. We also show that there is a size dependent bifurcation in the collective behavior in such systems.

PACS number(s): 05.45.−a

Maps are successful and widely used models of nonlinear and chaotic systems. All routes to chaos observed in high dimensional systems have been found in maps. Maps help us to understand the basic features and requirements of chaotic systems without going to higher dimensions and getting entangled in unnecessary details. They are not necessarily mathematical abstractions alone. There are cases like long delayed feedback in optical or hybrid optical systems for which maps can be rigorously derived from first principles [1]. Another example would be the case of periodically pulse-kicked oscillators [2]. For models of population dynamics, maps are natural mathematical systems of description [3]. Coupled map lattices (CML's) are an attempt to understand spatially extended nonlinear systems using building blocks that are well understood. We have seen a spurt of literature spanning various ideas in CML's. In the late 1980s, the one dimensional Euclidean lattice was extensively used to cover the phenomenological viewpoint [4]. Later discussions turned to globally coupled lattices, hierarchically coupled lattices, CML's on a fractal, random nonlocal couplings, or global inhomogeneous coupling, each of which was motivated by different physical systems and met varying degree of success in modeling those systems [5–11]. In this work we will investigate dynamics on connectivities different from the above, i.e., extensive interactions. In particular, we will study the dynamics on the recently introduced small-world lattices [12]. In the past we have seen extensive interactions in other work, namely, connectivities decaying as a power law with distance, and connectivities in which each site is connected to a range of neighbors that is a fraction of the total number of lattice sites [13].

Completely random and completely local connectivities have been studied in the past [8,14]. However, in many cases in real life, connections are known to be not completely random nor completely local but somewhere in between. This was modeled in an interesting work by Watts and Strogatz as the small-world model [12]. Examples are plenty: collaboration of movie stars, interactions of stockmarket brokers, and connectivity of internet web pages or neural nets. There are various studies on systems proposed by Watts and Strogatz. However, the dynamics of such connections has not been much studied. In this paper, we will study the dynamics of CML's with such connections, in particular, the synchronization on such networks. In this system the connections are local as well as nonlocal. It is clear that due to nonlocal interactions the range of interaction will keep growing as one increases the lattice size. Hence we call these interactions extensive.

In a previous work, we studied interactions that were extensive and local [13]. In that case we observed that in the thermodynamic limit synchronization is possible in a certain parameter regime if the number of sites connected to a given site is a finite *fraction* of the total number of sites. However, it is impossible if only a finite *number* of sites are connected. On the other hand, in the case of a small-world network, synchronization is always possible in the thermodynamic limit if a finite fraction of sites is connected, and even if a finite number of sites is connected to a given site, synchronization is possible in a certain parameter regime. Thus with nonlocal extensive interactions one is able to bring in synchronization in the presence of seemingly weaker conditions. As reflected in the behavior of the mean field, even the unsynchronized state shows a certain coherence for a large CML with finite nonlocal couplings. The reason for this qualitative difference could be that it is not possible to have extensive interaction for a finite number of local couplings, while interaction is extensive for nonlocal couplings.

Let us define a generic coupled map on a linear lattice of N sites with periodic boundary conditions. Let x_i be a variable associated with site i $(i=1,\dots N)$. The time evolution of x_i is given by

$$x_i(t+1)=\Sigma_{j=1}^{N} I_{i,j} f(x_j(t)). \qquad (1)$$

Here the connectivity matrix $I_{i,j}$ gives information on the connectivity and f is a nonlinear function. For example, the most explored connectivity is on a one dimensional lattice with nearest neighbor coupling, where $I_{i,i}=h_0$, $I_{i,i-1}=h_{-1}$, $I_{i,i+1}=h_1$, $i=1,\dots,N$, and other matrix elements are 0. By a synchronous state we mean a state in which $x_1=x_2=\cdots=x_N$. We note that, with the evolution of nearest neighbor coupling, if we start with a synchronized state the system stays synchronized. What we are interested in is the stability of the synchronized state, i.e., whether a system away from the synchronized state will reach the synchronized state asymptotically.

One would expect that networks will start behaving more coherently as a result of extensive interactions, even in the

*Electronic address: gade@helios.phy.ohiou.edu
†Electronic address: huck@phys.sinica.edu.tw

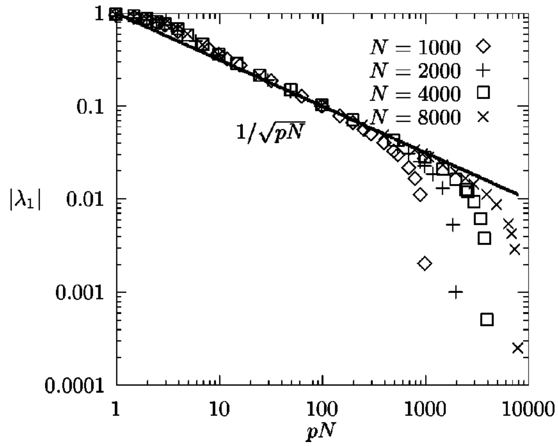

FIG. 1. The second largest eigenvalue as a function of pN (averaged over 50 configurations).

presence of temporal chaos. One of the most striking coherent modes is the synchronous mode. We will focus on the stability of this state in this work. Let us assume $\Sigma_{j=1}^{N} I_{i,j} = 1 \ \ \forall i$ so that a synchronous state exists. The stability of a temporally chaotic but spatially synchronized state can be determined analytically. This can be done by expanding the perturbation away from the synchronous state in terms of its eigenvectors. The only eigenmode that corresponds to the uniform state is $[1,1,\ldots,1]$. This is an eigenvector by construction if the synchronized state exists. It is easy to show that the condition for the stability of synchronous chaos is that only this eigenmode should survive and the rest should be damped [7,8,15]. The Jacobian of the synchronized state is directly related to the interaction matrix. Let λ_0 be the eigenvalue corresponding to this eigenmode $[1,1,\ldots,1]$ and λ_i, $i=1,2,\ldots,N-1$, represent the other $N-1$ eigenvalues of the interaction matrix ordered such that $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_{N-1}|$. Let λ be the Lyapunov exponent of the map f. It can be shown that the necessary condition for the stability of synchronous chaos is that only one eigenvalue $|\lambda_0 e^{\lambda}| > 1$ and the rest $|\lambda_i e^{\lambda}| < 1$ for $i=1,\ldots,N-1$ [6–8]. Thus the symmetries and topology of the interaction matrix are very important in determining the stability conditions. We will study the eigenvalue spectrum of the interaction matrices with extensive couplings and show that there is invariably a gap in the largest eigenvalue and the second largest eigenvalue in all of these cases.

In the original model by Watts and Strogatz [12], there is a possibility that the lattice can be broken into unconnected clusters. Here we study a slightly modified model by Newman and Watts [16]. In this model, we start with a regular one dimensional ring with N sites. Each site i is connected to its nearest neighbors $i+1$ and $i-1$ $[i+1 \in \mathrm{nbr}(i), i-1 \in \mathrm{nbr}(i)$ where nbr means "neighbor of]." In addition, we consider each of the N^2 possible pairs of sites $[(i,j), i=1,\ldots,N, j=1,\ldots,N]$ and with probability p we make a directed bond between them, i.e., j is the neighbor of $i[j \in \mathrm{nbr}(i)]$ with a probability p. We do not break any of the connections between two nearest neighbors. We allow coupling of the site with itself. We allow the site to be coupled to another site more than once and in such a case we count

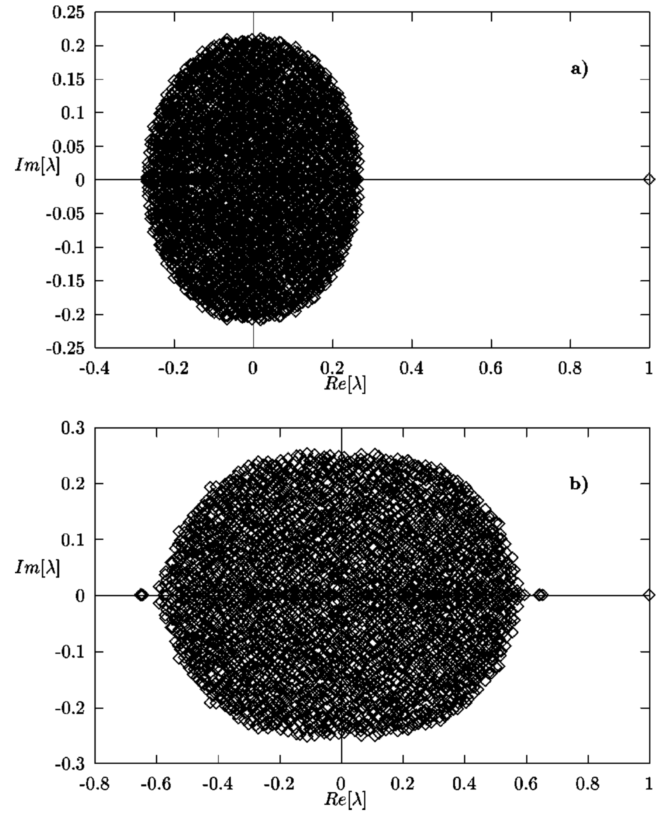

FIG. 2. (a) The eigenvalue spectrum for $pN=16$ in the complex plane. (b) The eigenvalue spectrum for $pN=4$ in the complex plane.

this bond more than once. Let $k(i)$ be the total number of connections including nearest neighbors for site i. We define the CML on this lattice as

$$x_i(t+1) = \frac{1}{k(i)} \Sigma_{j \in \mathrm{nbr}(i)} f(x_j(t)). \tag{2}$$

For example, if site $i=15$ has nonlocal neighbors 7, 3, and 7, i.e., 7 is chosen twice, $x_{15}(t+1) = (1/5)[f(x_{14}(t)) + f(x_{16}(t)) + f(x_7(t)) + f(x_3(t)) + f(x_7(t))]$.

As has been explained above, understanding the stability of the synchronized chaos in such a system demands analysis of its connectivity matrix and the question of interest is if the

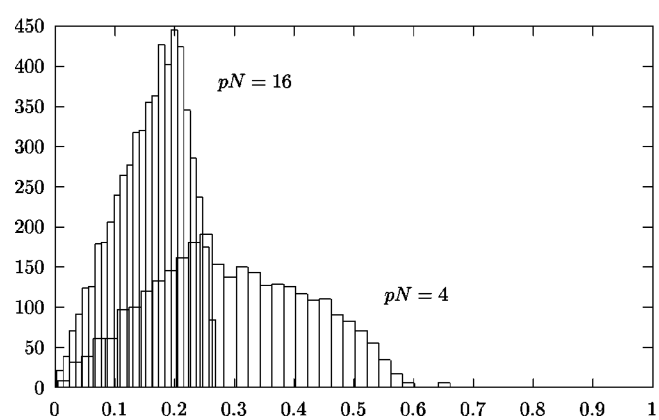

FIG. 3. Histograms of $|\lambda|$ for $pN=4$ and $pN=16$.

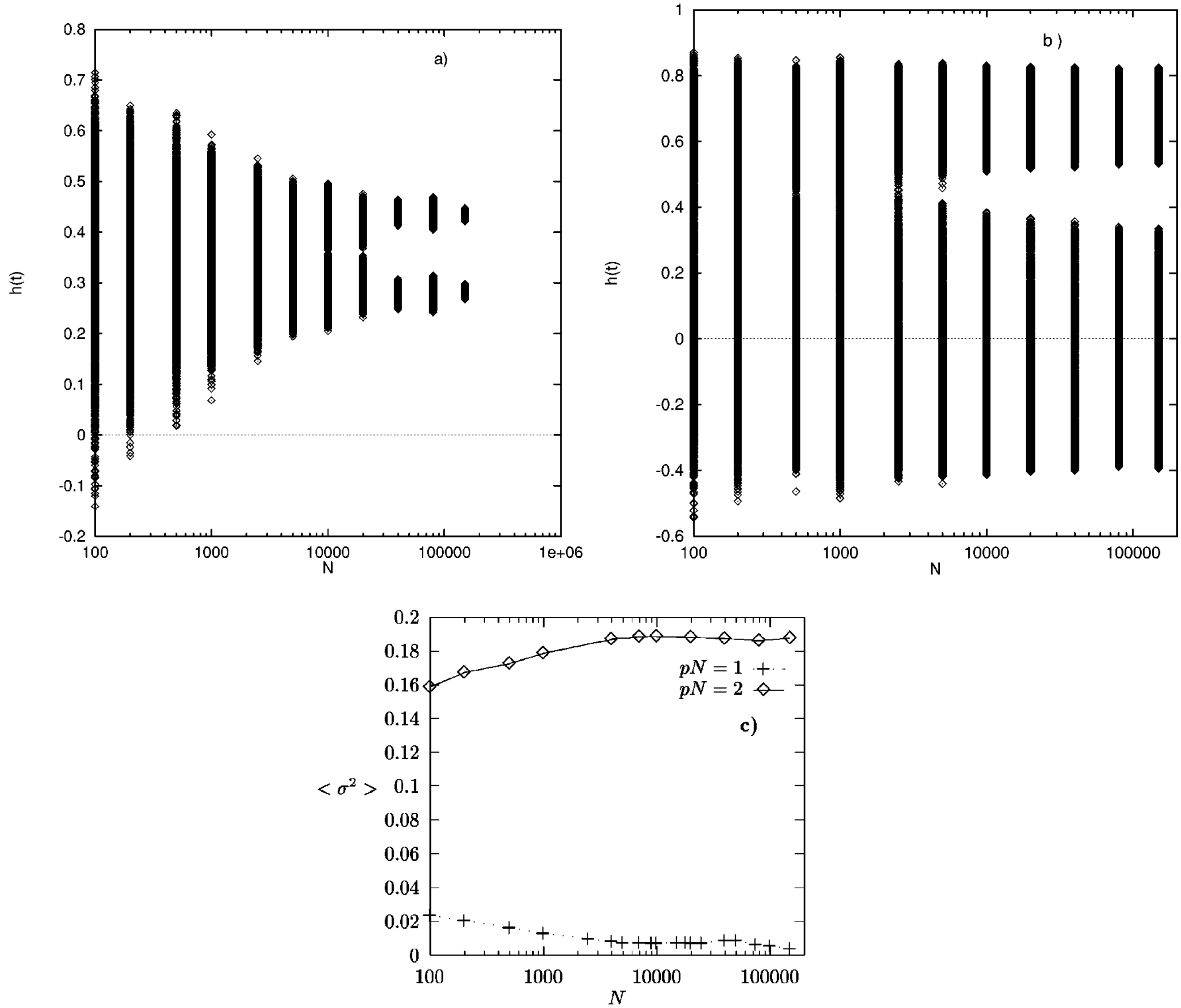

FIG. 4. (a) The bifurcation diagram of the mean field as a function of N for $pN=1$. (b) The bifurcation diagram of the mean field as a function of N for $pN=2$. (c) The variance of the mean field as a function of N for $pN=1$ and $pN=2$. The number of sites N is plotted on a logarithmic scale in all the figures.

eigenvalues have a gap. We have diagonalized these matrices numerically and they show a clear gap. The spectrum seems to be a function of pN instead of p or N. This means that the number of nonlocal connections of each site determines the spectrum. Previously, we analyzed coupled maps on completely random networks in which each site was coupled to k neighbors chosen randomly. The second largest eigenvalue in the spectrum varied as $1/\sqrt{k}$ [8]. For large pN, i.e., a large number of nonlocal connections, we expect a similar behavior since each lattice point will have pN nonlocal connections on average. Figure 1 shows the second largest eigenvalue as a function of pN. Each point has an average of 50 configurations. The anticipated trend for large pN is better for large N. Figure 2 shows the eigenvalues for this matrix in the complex plane. The picture is very similar to that observed in fully nonlocal connectivity. Probably this has to do with the large number of nonlocal connections for this case. Figure 3 shows the histogram of the distribution of absolute values of the eigenvalues. This is most probably a single peaked distribution.

The eigenvalues in the limits $p=0$ and $p=1$ are simple. In the former limit the absolute value of the second largest eigenvalue tends to 1, while in the latter limit it tends to $2/(N+2)$. In the region in between, we expect it to behave as $1/\sqrt{pN}$. We observe that the eigenvalues smoothly interpolate between these expected behaviors.

The fact that the number of nonlocal connections matters and not their fraction is quite interesting. In a previous publication [13], we found that there is no gap in the eigenvalue spectrum in the thermodynamic limit for a fixed number of local connections for a lattice site. The gap exists only if a fixed fraction of sites is connected. However, in the case of nonlocal connectivity even one or two nonlocal connections are enough to cause a gap in the spectrum, allowing the possibility of synchronous chaos. Furthermore, if a fixed fraction of sites are connected nonlocally, synchronous chaos is *always* stable in the thermodynamic limit. For a fixed $p>0$, synchronous chaos is always a stable attractor in the thermodynamic limit since the second largest eigenvalue (which varies as $1/\sqrt{pN}$) of the connectivity matrix will tend to zero as $N \rightarrow \infty$.

The gap in the eigenvalue spectrum means that synchronization is indeed possible on such lattices and is more and more likely as the number of nonlocal connections increases. This spectrum is related to the spectrum of the Laplacian operator. The spectrum of the Laplacian operator in the original scheme of the small-world lattice by Watts and Strogatz has been studied from the viewpoint of localization, diffusion, and dispersion relations. Thus study of the interaction matrix is useful for purposes other than synchronization. In these systems, there is no true gap in the spectrum, unlike the case in our system [17]. Our study on a modified small-world lattice in which nonlocal connections do not come into being at the expense of local connections should also shed light on these questions.

Whenever the above condition, i.e., $\lambda_i e^{\lambda} < 1$ for $i = 1, \ldots, N-1$ and $\lambda_0 e^{\lambda} > 1$, is satisfied, it implies the linear stability of the synchronized state, but it does not tell much about its basin of attraction. However, we observe synchronization from a wide range of initial conditions. This is unlike the case of a globally coupled array where, despite linear stability, the system away from the synchronized state may not reach a synchronized state easily because of attractor crowding [18]. A globally coupled system has high symmetry. In fact, if there is an attractor in addition to the synchronized state, there will be $N!$ equivalent attractors crowding the phase space. The system of the present paper has very low symmetries due to randomness and thus it is easier to reach the synchronized state.

The collective behavior of CML's in various dimensions has been studied extensively in recent years. It seems that an interesting nontrivial collective behavior is obtained in higher dimensions in CML's even in the presence of local chaos. There have been detailed studies of the behavior of the mean field in CML's in globally coupled maps [5], coupled maps and cellular automata in higher dimensions [10], and coupled maps with random nonlocal couplings [8]. We will study the behavior of the mean field on a small-world lattice for a given number of average nonlocal connections pN, since for constant p one will always get synchronization in the thermodynamic limit. We studied the behavior of the mean field $h(t) = (1/N)\sum_{i=1}^{N} x_i(t)$ as a function of N for constant pN. We chose the local map $f(x)$ as the logistic map at fully developed chaos, $f(x) = 1 - 2x^2$.

We studied this behavior for two values $pN = 1$ and $pN = 2$, i.e., the cases of one nonlocal coupling per site and two nonlocal couplings per site on average. Since there are two local coupling at each site, we felt this is representative of what happens when nonlocal couplings are weaker than local couplings and when they are as strong as local couplings. We find that the mean field develops a two-band type of behavior for larger lattices in both cases. However, for $pN = 2$, i.e., when the nonlocal connections are as strong as the local connections, a strong collective behavior emerges even in the absence of synchronization. Figure 4(a) and Fig. 4(b) show the ''bifurcation diagram'' for the mean field as a function of the total number of sites for $pN = 1$ and for $pN = 2$. There are differences in behaviors of individual configurations due to randomness. We have shown representative configurations, i.e., those that have behavior seen in the majority of configurations. For $pN = 2$, the difference between the behavior of the mean field in different configurations is very small for large lattices ($N > 20\,000$). However, for $pN = 1$ the different configurations continue to be quite different. In both cases, there is a range of values $N_1 < N < N_2$ such that one may observe either a two-band or one-band structure in the mean field depending on the configuration. We find that $N_1 \sim 1000$ and $N_2 \sim 40\,000$ for $pN = 1$ and $N_1 \sim 1500$ and $N_2 \sim 10\,500$ for $pN = 2$. For $N \leqslant N_1$ we find exclusively one-band structure while for values of $N \geqslant N_2$ we find exclusively two-band structure. The ''bifurcation'' from one-band structure to two-band structure is not well defined, which is understandable since this is a random system and at smaller lattice sizes individual configurations may vary a lot. We tried to study the variance of the mean field $\sigma^2 = \langle h(t)^2 \rangle - \langle h(t) \rangle^2$, and we used the mean value of the variance over several configurations $\langle \sigma^2 \rangle$ to quantify the uncorrelatedness of individual elements. Given the structure in the mean field, the variance does not obey the law of large numbers and does not approach zero in the thermodynamic limit. If all the lattice elements were unrelated, we would expect the mean field to converge to a constant value. In fact, for $pN = 2$, $\langle \sigma^2 \rangle$ grows and saturates. For $pN = 1$, however, there is a decrease in $\langle \sigma^2 \rangle$ that is slower than $1/N$. Figure 4(c) shows $\langle \sigma^2 \rangle$ as a function of N for $pN = 1$ and for $pN = 2$. The averaging is performed over at least 150 configurations. The growth of the variance of the mean field as a function of N is rather strange and needs further investigation. The mean field indeed has a much more coherent structure for $pN = 2$ than for $pN = 1$. Figure 5 shows the return map of the mean field for $pN = 1$ and $pN = 2$ for $N = 1.5 \times 10^5$. For $pN = 2$, one can clearly see a structure in the return map of the mean field which is very similar to the original logistic map.

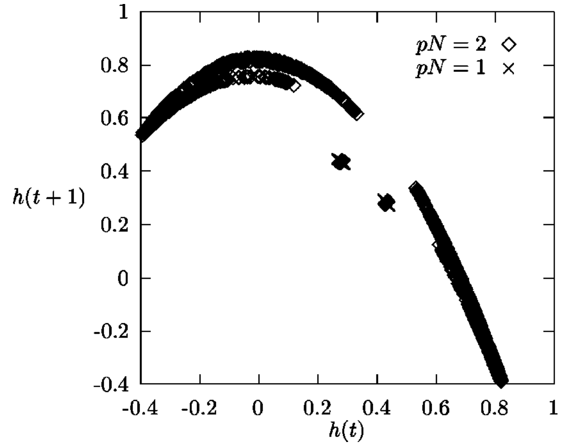

FIG. 5. The return map for the mean field, i.e., $h(t+1)$ as a function of $h(t)$ for $pN = 1$ and $pN = 2$ for $N = 1.5 \times 10^5$.

Extensive interactions arise in several physical situations in nature. In general, they lead to higher coherence in spatially extended systems. In this work, we have explored the stability of synchronous chaos in coupled map lattices with small-world connectivity, which is an extensive connectivity in the sense that the range of interaction keeps growing with the lattice size. We have shown that in this case the eigenvalue spectrum of the interaction matrix has a gap and thus synchronous chaos is possible even in the thermodynamic limit. Small-world connectivity is conjectured to exist in systems as diverse as power grid connections and neural nets. We have showed that randomness and extensivity of coupling play important roles in reaching coherence in this case. We have also shown that, even in cases where synchronization is not reached, dynamical behavior can be highly coherent in small world networks.

This work was supported by the National Science Council of the Republic of China (Taiwan) under Grant No. NSC 89-2112-M001-005.

[1] See, for example, B. Mensour and A. Longtin, Phys. Rev. E **58**, 410 (1998).
[2] See, for example, C. R. de Oliveira and P. S. Goncalves, Phys. Rev. E **56**, 4868 (1997).
[3] See, for example, R. M. May, Nature (London) **261**, 151 (1976).
[4] J. P. Crutchfield and K. Kaneko, in *Directions in Chaos*, edited by Hao Bai-Lin (World Scientific, Singapore, 1987), and references therein.
[5] K. Kaneko, Phys. Rev. Lett. **65**, 1391 (1990), and references therein.
[6] P. M. Gade and R. E. Amritkar, Phys. Rev. E **47**, 143 (1993).
[7] P. M. Gade, H. Cerdeira, and R. Ramaswamy, Phys. Rev. E **52**, 2478 (1995).
[8] P. M. Gade, Phys. Rev. E **54**, 64 (1996).
[9] S. Raghavachari and J. A. Glazier, Phys. Rev. Lett. **74**, 3297 (1995).
[10] H. Chate and P. Manneville, Prog. Theor. Phys. **87**, 1 (1992).
[11] D. H. Zanette, Europhys. Lett. **45**, 424 (2000).
[12] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[13] P. M. Gade and C.-K. Hu, Phys. Rev. E **60**, 4966 (1999).
[14] S. C. Manrubia and A. S. Mikhailov, Phys. Rev. E **60**, 1579 (1999).
[15] See also L. M. Pecora and T. L. Caroll, Phys. Rev. Lett. **80**, 2109 (1998) for a similar treatment of oscillator arrays. For recent work, see K. S. Fink *et al.*, Phys. Rev. E **61**, 5080 (2000).
[16] M. E. J. Newman and D. J. Watts, e-print cond-mat/9904419.
[17] R. Monasson, e-print cond-mat/9903347.
[18] K. Y. Tsang and K. Wiesenfeld, Appl. Phys. Lett. **56**, 495 (1990).